J Math Chem (2010) 47:276–294
DOI 10.1007/s10910-009-9567-8

ORIGINAL PAPER

Global behaviors of Monod type chemostat model with nutrient recycling and impulsive input

Zhidong Teng · Rong Gao · Mehbuba Rehim · Kai Wang

Received: 18 February 2009 / Accepted: 29 June 2009 / Published online: 26 August 2009

Abstract In this paper, we consider the global behaviors of Monod type chemostat model with nutrient recycling and impulsive input. By introducing a new study method, the sufficient and necessary conditions on the permanence and extinction of the microorganisms are obtained. Furthermore, by using the Liapunov function method, the sufficient condition on the global attractivity of the system is established. Lastly, an example is given, the numerical simulation shows that if only the system is permanent, then it also is globally attractive.

Keywords Chemostat · Nutrient recycling · Impulsive input · Permanence · Extinction · Global attractivity

1 Introduction

The chemostat is a very important apparatus used to study the growth of microorganisms in a continuous cultured environment in a laboratory. It may be viewed as a laboratory model of a simple lake with continuous stirring. Chemostat models have attracted widely the attention of the scientific community, since they have a wide range of applications, for example, waste water treatment, production by genetically altered organisms (like production of insulin), etc. The growth in a chemostat is described by the systems of ordinary differential equations or functional differential equations. Generally, in a chemostat the loss or death of biomass is attributed to the washout and the nutrient and its consumer are washed out of the system at a very high rate. However, when we try to model a natural lake system the washout rate tends to be low. When the washout rate is low, the dead biomass (death may be natural) remains in the

Z. Teng (✉) · R. Gao · M. Rehim · K. Wang
College of Mathematics and System Sciences, Xinjiang University,
830046 Urumqi, People's Republic of China
e-mail: zhidong@xju.edu.cn

system and there is every possibility for the bacterial decomposition of dead biomass resulting in the regeneration of nutrient. Thus, we may introduce a recycling of dead biomass as nutrient. The chemostat models with nutrient recycling have been extensively investigated by many researchers. The studied main subjects are the persistence, permanence and extinction of microorganisms, global stability and the existence of periodic oscillation of the systems, etc. Many important and interesting results can be found in articles [1–7,11–18] and the references cited therein.

In recent years, many scholars pointed out that it was necessary and important to consider biological models with periodic perturbations, since these models might be quite naturally exposed in many real world phenomena. In fact, almost perturbations occur in a more-or-less periodic fashion. However, there are some other perturbations such as fires, floods, and drainage of sewage which are not suitable to be considered continually. These perturbations bring sudden changes to the system and often be characterized mathematically in the form of impulses. Systems with sudden perturbations are involving in impulsive differential equations. The chemostat models with impulsive input perturbation have been studied in many articles, see [8–10,19–26] and the references cited therein, where many important and interesting results on the persistence, permanence and extinction of microorganisms, global stability, the existence of periodic oscillation and dynamical complexity of the systems are discussed. In particular, in [19], the following Monod type chemostat model with impulsive input is discussed

$$
\begin{aligned}
S'(t) &= -QS(t) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))}, && t \neq nT,\\
x'(t) &= x(t)\frac{\mu_m S(t)x(t)}{K_m + S(t)} - Q), && t \neq nT,\\
S(t^+) &= S(t) + DS^0, && t = nT,\\
x(t^+) &= x(t), && t = nT.
\end{aligned}
$$

The sufficient conditions on the permanence and extinction of the system are established, see Theorems 3.1–3.3 in [19].

However, we observe that the research on the chemostat model with impulsive perturbations and nutrient recycling is not too much yet. Therefore, as a result, in this paper we consider the following Monod type chemostat model with nutrient recycling and impulsive input

$$
\begin{aligned}
S'(t) &= -DS(t) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))} + b\gamma x(t), && t \neq nT,\\
x'(t) &= \frac{\mu_m S(t)x(t)}{K_m + S(t)} - (D+\gamma)x(t), && t \neq nT,\\
S(t^+) &= S(t) + DS^0, && t = nT,\\
x(t^+) &= x(t), && t = nT,
\end{aligned}
\tag{1.1}
$$

For system (1.1) we will investigate the permanence, extinction and the global asymptotic stability. we will establish the sufficient and necessary conditions for the permanence and extinction and the sufficient condition for the global asymptotic stability. We also will give an example and numerical simulation to show when the sufficient condition on the global asymptotic stability does not hold, the system still may be globally asymptotically stable.

This paper is organized as follows. In the following section we will give several useful lemmas. In Sect. 3 we will state and prove our main results on the extinction, permanence and global asymptotic stability. In Sect. 4, we will discuss an example and give the numerical simulation.

2 Preliminaries

In system (1.1), $t \in R_+ = [0, \infty)$, $n \in N$, N is the set of nonnegative integers, $S(t)$ denote the limiting nutrient concentration at time t, $x(t)$ denote the plankton concentration at time t. S^0 is the input concentration of the limiting nutrient, T is the period of pulsing, D is the washout rate, DS^0 denote the input concentration of the limiting substrate per unit of time, b is the fraction of the nutrient recycled by bacterial decomposition of the dead plankton, δ denotes the ratio of microorganism produced to the mass of the substrate consumed. Obviously, we have $0 \le b \le 1$ and $\frac{1}{\delta} \ge 1$, γ is the death rate of plankton, so $D + \gamma$ represents the total loss rate of the plankton. In this paper, we always assume that all parameters in system (1.1) are positive constants.

In addition, in system (1.1), $S(nT^+) = \lim_{t\to nT^+} S(t)$, $x(nT^+) = \lim_{t\to nT^+} x(t)$, $S(t)$ is assumed to be left continuous at $t = nT$, that is, $S(nT) = \lim_{t\to nT^-} S(t)$, and $x(t)$ is assumed to be continuous at $t = nT$.

Let $R^2_+ = \{(x_1, x_2) \in R^2 : x_1 > 0,\ x_2 > 0\}$. On the positivity of solutions for system (1.1) we have the following result.

Lemma 2.1 *For any $(S_0, x_0) \in R^2_+$, the solution $(S(t), x(t))$ of system (1.1) with initial condition $S(0^+) = S_0$ and $x(0) = x_0$ is positive, that is, $S(t) > 0$ and $x(t) > 0$ for any $t > 0$.*

The proof of Lemma 2.1 is simple, we hence omit it here.

We consider the following impulsive differential equation

$$\begin{aligned} u'(t) &= -d_1 u(t), \quad t \ne nT,\ t \in R_+, \\ u(t^+) &= u(t) + d_2, \quad t = nT,\ n \in N. \end{aligned} \tag{2.1}$$

We have the following result.

Lemma 2.2 *Assume that T and d_i $(i = 1, 2)$ are positive constants, then Eq. 2.1 has a positive periodic solution*

$$u^*(t) = \frac{d_2 e^{-d_1(t-nT)}}{1 - e^{-d_1 T}} \quad \textit{for all} \quad t \in (nT, (n+1)T],\ n \in N,$$

which is globally uniformly attractive, that is, for any constants $\varepsilon > 0$ and $M > 0$ there is a $T = T(\varepsilon, M) > 0$ such that for any $t_0 \in R_+$ and $u_0 \in R$ with $|u_0| \le M$ one has

$$|u(t, t_0, u_0) - u^*(t)| < \varepsilon \quad \textit{for all} \quad t \geq t_0 + T,$$

where $u(t, t_0, u_0)$ *is the solution of Eq.* 2.1 *with initial condition* $u(t_0) = u_0$.

Proof Calculating the solution $u(t)$ of Eq. 2.1 with initial condition $u(0) = u_0$ on $t \in [0,\ T]$, we have

$$u(t) = u_0 e^{-d_1 t} \quad \text{for all} \quad t \in [0,\ T)$$

and

$$u(T^+) = u_0 e^{-d_1 T} + d_2.$$

Let $u(0) = u(T^+)$, then we have $u_0 = \frac{d_2}{1-e^{-d_1 T}}$. Therefore, Eq. 2.1 has the positive T-periodic solution as follows

$$u^*(t) = \frac{d_2 e^{-d_1(t-nT)}}{1-e^{-d_1 T}} \quad \text{for all} \quad t \in (nT, (n+1)T],\ n \in N.$$

Obviously, $u^*(t) \leq \frac{d_2}{1-e^{-d_1 T}}$ for all $t \in R_+$.

For any constants $M > 0$, let $u(t, t_0, u_0)$ is the solution of Eq. 2.1 with initial condition $u(t_0) = u_0$, where $t_0 \in R_+$ and $u_0 \in R$ with $|u_0| \leq M$. Let $v(t) = u(t, t_0, u_0) - u^*(t)$ then for any $t \geq t_0$ we have

$$\begin{aligned} v'(t) &= -d_1 v(t), \quad && t \neq nT, \\ v(t^+) &= v(t), && t = nT. \end{aligned}$$

Hence,

$$v(t) = (u_0 - u^*(t_0)) e^{-d_1(t-t_0)} \quad \text{for all} \quad t \geq t_0.$$

Consequently,

$$|u(t, t_0, u_0) - u^*(t)| \leq \left(M + \frac{d_2}{1-e^{-d_1 T}}\right) e^{-d_1(t-t_0)} \quad \text{for all} \quad t \geq t_0.$$

For any constants $\varepsilon > 0$, choosing

$$T = T(\varepsilon, M) = \frac{1}{d_1}\left(\ln\left(M + \frac{d_2}{1-e^{-d_1 T}}\right) - \ln \varepsilon\right),$$

then we finally have

$$|u(t, t_0, u_0) - u^*(t)| < \varepsilon \quad \text{for all} \quad t \geq t_0 + T.$$

This completes the proof of Lemma 2.2. □

Further, on the ultimate boundedness of all positive solutions of system (1.1) we have the following result.

Lemma 2.3 *For any solution* $(S(t), x(t))$ *of system* (1.1) *with initial value* $(S(0^+), x(0)) \in R^2_+$, *we have*

$$\limsup_{t\to\infty} S(t) \le M, \quad \limsup_{t\to\infty} x(t) \le \delta M,$$

where $M = \frac{DS^0}{1-e^{-DT}}$.

Proof Let $(S(t), x(t))$ be any solution of system (1.1) with initial value $(S(0^+), x(0)) \in R^2_+$. Define

$$V(t) = S(t) + \frac{1}{\delta}x(t).$$

Then we have

$$\begin{aligned} V'(t) &= -DS(t) - \frac{D}{\delta}x(t) + b\gamma x(t) - \frac{\gamma}{\delta}x(t) \\ &= -DV(t) + \gamma\left(b - \frac{1}{\delta}\right)x(t) \\ &\le -DV(t), \quad t \ne nT, \ n \in N. \\ V(t^+) &= V(t) + DS^0, \quad t = nT. \end{aligned}$$

From the comparison theorem of impulse differential equations, we have $V(t) \le u(t)$ for all $t \ge 0$, where $u(t)$ is the solution of the following comparison equation

$$\begin{aligned} u'(t) &= -Du(t), \qquad t \ne nT, \ n \in N. \\ u(t^+) &= u(t) + DS^0, \quad t = nT. \end{aligned} \tag{2.2}$$

with initial condition $u(0^+) = V(0^+)$. From Lemma (2.3), Eq. 2.2 has a unique globally uniformly attractive positive T-periodic solution

$$u^*(t) = \frac{DS^0 e^{-D(t-nT)}}{1 - e^{-DT}} \quad \text{for all} \quad t \in (nT, (n+1)T], \ n \in N. \tag{2.3}$$

Hence, we have $u(t) \to u^*(t)$ as $t \to \infty$. From this, we finally have

$$\limsup_{t\to\infty} V(t) \le \limsup_{t\to\infty} u^*(t) \le \frac{DS^0}{1 - e^{-DT}}.$$

This completes the proof of Lemma 2.3. □

3 Main results

For system (1.1), if we choose $x(t) \equiv 0$ then system (1.1) becomes to the following subsystem

$$\begin{aligned} S'(t) &= -DS(t), & t \neq nT, \\ S(t^+) &= S(t) + DS^0, & t = nT, \end{aligned} \tag{3.1}$$

System (3.1) has a unique globally uniformly attractive positive T-periodic solution $u^*(t)$ which is given in (2.3). Hence, system (1.1) has a T-periodic solution $(u^*(t), 0)$ at which microorganism culture fails. On the global attractivity of $(u^*(t), 0)$ for system (1.1), we have the following result.

Theorem 3.1 *Suppose*

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D + \gamma) \right) dt \leq 0. \tag{3.2}$$

Then periodic solution $(u^*(t), 0)$ *of system* (1.1) *is globally attractive.*

Proof Let $(S(t), x(t))$ be any positive solution of system (1.1). Define function as follows

$$V(t) = S(t) + \frac{1}{\delta} x(t),$$

then similar to the proof of Lemma 2.3 we obtain $V(t) \leq u(t)$ for all $t \geq 0$, where $u(t)$ is the solution of Eq. 2.2 with initial value $u(0^+) = V(0^+)$, and $u(t) \to u^*(t)$ as $t \to \infty$. Hence, there exists a function $\alpha(t) : R_+ \to R$ satisfying $\alpha(t) \to 0$ as $t \to \infty$ such that

$$V(t) \leq u(t) = u^*(t) + \alpha(t)$$

for all $t \geq 0$. From the definition of $V(t)$ we further have

$$S(t) \leq u^*(t) + \alpha(t) - \frac{1}{\delta} x(t).$$

From the second equation of the system (1.1) we obtain

$$x'(t) \leq x(t) \left(\frac{\mu_m \left(u^*(t) + \alpha(t) - \frac{1}{\delta} x(t) \right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta} x(t)} - (D + \gamma) \right). \tag{3.3}$$

From condition (3.2) we obtain for any $\varepsilon_0 > 0$

$$\int_0^T \frac{\mu_m\left(u^*(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) - \frac{1}{\delta}\varepsilon_0}\mathrm{d}t - (D+\gamma)T < 0.$$

Since $\lim_{t\to\infty}\alpha(t) = 0$, we can obtain

$$\lim_{t\to\infty}\left(\int_t^{t+T} \frac{\mu_m\left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0}\mathrm{d}t - (D+\gamma)T\right)$$
$$= \int_0^T \frac{\mu_m\left(u^*(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) - \frac{1}{\delta}\varepsilon_0}\mathrm{d}t - (D+\gamma)T < 0.$$

Hence, there exist constants $\eta > 0$ and $T_0 > 0$ such that when $t \geq T_0$

$$\int_t^{t+T} \frac{\mu_m\left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0}\mathrm{d}t - (D+\gamma)T \leq -\eta \tag{3.4}$$

and $|\alpha(t)| < 1$.

If $x(t) \geq \varepsilon_0$ for all $t \geq T_0$, then from (3.3) we obtain

$$x'(t) \leq x(t)\left(\frac{\mu_m\left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D+\gamma)\right). \tag{3.5}$$

For any $t \geq T_0$, we choose an integer $p \geq 0$ such that $t \in [T_0 + pT_0 + (p+1)T)$, then integrating (3.5) from T_0 to t, from (3.4) we can obtain

$$x(t) \leq x(T_0)\exp\left\{\int_{T_0}^{t}\left(\frac{\mu_m\left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D+\gamma)\right)\mathrm{d}t\right\}$$
$$= x(T_0)\exp\left\{\left(\int_{T_0}^{T_0+pT} + \int_{T_0+pT}^{t}\right)\left(\frac{\mu_m\left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D+\gamma)\right)\mathrm{d}t\right\}$$
$$\leq x(T_0)\exp(-\eta p)\exp\left\{\int_{T_0+pT}^{t}\left(\frac{\mu_m\left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D+\gamma)\right)\mathrm{d}t\right\}$$
$$\leq x(T_0)\exp(-\eta p)\exp(\sigma_0 T), \tag{3.6}$$

where

$$\sigma_0 = \frac{\mu_m \left(M + 1 - \frac{1}{\delta}\varepsilon_0\right)}{K_m + M + 1 - \frac{1}{\delta}\varepsilon_0} - (D + \gamma)$$

and constant M is given in Lemma 2.3. Since $p \to \infty$ as $t \to \infty$, we obtain $x(t) \to 0$ as $t \to \infty$ from (3.6), which leads to a contradiction. Hence, there is a $t^* \geq T_0$ such that $x(t^*) < \varepsilon_0$.

Now, we claim that there exists a constat $M_0 > 1$ such that $x(t) \leq \varepsilon_0 M_0$ for all $t \geq t^*$. In fact, if there exists a $t_1 > t^*$ such that $x(t_1) > \varepsilon_0 M_0$, then there exists a $t_2 \in (t^*, t_1)$ such that $x(t_2) = \varepsilon_0$ and $x(t) > \varepsilon_0$ for $t \in (t_2, t_1)$. Choose an integer $p \geq 0$ such that $t_1 \in [t_2 + pT, t_2 + (p+1)T)$. Since for any $t \in (t_2,\ t_1)$

$$x'(t) \leq x(t)\left(\frac{\mu_m \left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D + \gamma)\right),$$

integrating this inequality from t_2 to t_1, from (3.4) we can obtain

$$\begin{aligned} x(t_1) &\leq x(t_2)\exp\left\{\int_{t_2}^{t_1}\left(\frac{\mu_m \left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D + \gamma)\right)\mathrm{d}t\right\} \\ &= x(t_2)\exp\left\{\left(\int_{t_2}^{t_2+pT} + \int_{t_2+pT}^{t_1}\right)\left(\frac{\mu_m \left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D + \gamma)\right)\mathrm{d}t\right\} \\ &\leq x(t_2)\exp(-\eta p)\exp\left\{\int_{t_2+pT}^{t_1}\left(\frac{\mu_m \left(u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0\right)}{K_m + u^*(t) + \alpha(t) - \frac{1}{\delta}\varepsilon_0} - (D + \gamma)\right)\mathrm{d}t\right\} \\ &\leq \varepsilon_0 \exp(\sigma_0 T). \end{aligned} \tag{3.7}$$

Obviously, choose constant $M_0 = \exp(\sigma_0 T)$, then from (3.7) we obtain a contradiction. Hence, we have $x(t) \leq \varepsilon_0 M_0$ for all $t \geq t^*$. Since ε_0 is arbitrary, we finally have

$$\lim_{t\to\infty} x(t) = 0.$$

This completes the proof of Theorem 3.1. □

Next, we discuss the permanence of system (1.1), we have the following result.

Theorem 3.2 *System* (1.1) *is permanent, if*

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D + \gamma)\right)\mathrm{d}t > 0.$$

Proof Let $(S(t), x(t))$ be any solution of system (1.1) with initial value $(S(0^+), x(0^+)) \in R_+^2$. From Lemma 2.3, without loss of generality, we can assume $S(t) \le M$ and $x(t) \le M$ for all $t \ge 0$. From the first equation of system (1.1) we obtain

$$\begin{aligned} S'(t) &\ge -DS(t) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))} \\ &> -DS(t) - \frac{\mu_m M S(t)}{\delta K_m} \\ &= -\left(D + \frac{\mu_m M}{\delta K_m}\right) S(t), \quad t \ne nT, \\ S(t^+) &= S(t) + DS^0, \quad t = nT. \end{aligned}$$

Using Lemma 2.2 and the comparison theorem of impulsive differential equation, we obtain $S(t) \ge v(t)$ for all $t \ge 0$, where $v(t)$ is the solution of the following impulsive equation

$$\begin{aligned} v'(t) &= -\left(D + \frac{\mu_m M}{\delta K_m}\right) v(t), \quad t \ne nT, \\ v(t^+) &= v(t) + DS^0, \qquad\qquad t = nT \end{aligned}$$

with initial condition $v(0^+) = S(0^+)$. Further from Lemma 2.2, we have

$$\lim_{t\to\infty} v(t) = v^*(t),$$

where

$$v^*(t) = \frac{DS^0 \exp\left\{-\left(D + \frac{\mu_m M}{\delta K_m}\right)(t - nT)\right\}}{1 - \exp\left\{-\left(D + \frac{\mu_m M}{\delta K_m}\right)T\right\}}.$$

Therefore, we further obtain

$$\begin{aligned} \liminf_{t\to\infty} S(t) &\ge \liminf_{t\to\infty} v(t) = \liminf_{t\to\infty} v^*(t) \\ &\ge \frac{DS^0 \exp\left(-\left(D + \frac{\mu_m M}{\delta K_m}\right)T\right)}{1 - \exp\left(-\left(D + \frac{\mu_m M}{\delta K_m}\right)T\right)}. \end{aligned}$$

This shows that $S(t)$ in system (1.1) is permanent.

Next, we prove that there exists a constant $m_2 > 0$ such that

$$\liminf_{t\to\infty} x(t) \ge m_2$$

for any positive solution $(S(t), x(t))$ of system (1.1). From

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D+\gamma)\right) \mathrm{d}t > 0,$$

we can choose a constant $\varepsilon_0 > 0$ small enough such that

$$\sigma \triangleq \int_0^T \left(\frac{\mu_m (u^*(t) - \varepsilon_0)}{K_m + u^*(t) - \varepsilon_0} - (D+\gamma)\right) \mathrm{d}t > 0.$$

Consider the following auxiliary impulsive equation

$$\begin{aligned} y'(t) &= -y\left(D + \tfrac{m_3\mu_m}{\delta K_m}\right), \quad t \neq nT \\ y(t^+) &= y(t) + DS^0, \qquad t = nT, \end{aligned} \tag{3.8}$$

from Lemma 2.2, Eq. 3.8 have a globally uniformly attractive T-periodic positive solution

$$y^*(t) = \frac{DS^0 \exp\left\{-\left(D + \frac{m_3\mu_m}{\delta K_m}\right)(t - nT)\right\}}{1 - \exp\left\{-\left(D + \frac{m_3\mu_m}{\delta K_m}\right)T\right\}}, \quad t \in (nT, (n+1)T], \quad n \in N.$$

Since $\lim_{m_3 \to 0} y^*(t) = u^*(t)$, for above $\varepsilon_0 > 0$ there is a $m_3 > 0$ such that

$$y^*(t) \geq u^*(t) - \frac{\varepsilon_0}{2} \tag{3.9}$$

for $t \geq 0$. Further, for above $\varepsilon_0 > 0$ and $M > 0$, there is a $T_0 = T_0(\varepsilon_2, M) > 0$ such that for any $t_0 \geq 0$ and $0 \leq y_0 \leq M$ we have

$$|y(t, t_0, y_0) - y^*(t)| < \frac{\varepsilon_0}{2} \tag{3.10}$$

for all $t \geq t_0 + T_0$, where $y(t, t_0, y_0)$ is the solution of Eq. 3.8 with initial condition $y(t_0^*) = y_0$

For any $t_0 \geq 0$, if $x(t) \leq m_3$ for all $t \geq t_0$, then from system (1.1) we have

$$\begin{aligned} S'(t) &\geq -S(t)\left(D + \frac{m_3\mu_m}{\delta K_m}\right), \quad t \neq nT, \\ S(t^+) &= S(t) + DS^0, \qquad t = nT, \ n \in N \end{aligned}$$

for all $t \geq t_0$. By the comparison theorem of impulse differential equation, we have $S(t) \geq y(t)$ for $t \geq t_0$, where $y(t)$ is the solution of Eq. 3.8 with initial condition

$y(t_0^+) = S(t_0^+)$. Directly from (3.10) we obtain

$$|y(t) - y^*(t)| < \frac{\varepsilon_0}{2} \quad \text{for all} \quad t \geq t_0 + T_0.$$

Hence, from (3.9) we further have

$$S(t) \geq S^*(t) - \varepsilon_0 \quad \text{for all} \quad t \geq t_0 + T_0.$$

From the second equation of system (1.1) we have

$$x'(t) \geq x(t)\left(\frac{\mu_m(u^*(t) - \varepsilon_0)}{K_m + u^*(t) - \varepsilon_0} - (D + \gamma)\right) \tag{3.11}$$

for all $t \geq t_0 + T_0$.

Let $n_0 \in N$ such that $n_0 T > t_0 + T_0$. Integrating (3.11) on $(nT, (n+1)T]$ for all $n \geq n_0$, we have

$$\begin{aligned} x((n+1)T) &\geq x(nT^+) \exp\left\{\int\limits_{nT}^{(n+1)T} \left(\frac{\mu_m(u^*(t) - \varepsilon_0)}{K_m + u^*(t) - \varepsilon_0} - (D+\gamma)\right) \mathrm{d}t\right\} \\ &= x(nT)e^{\sigma}. \end{aligned}$$

Hence, $x((n_0 + k)T) \geq x(n_0 T)e^{k\sigma}$ for all $k \geq 0$. Consequently, we have $\lim_{t\to\infty} x((n_0+k)T) = \infty$, which leads to a contradiction. Therefore, there exists a $t_1 > t_0 + T_0$ such that $x(t_1) \geq m_3$.

If $x(t) \geq m_3$ for all $t \geq t_1$, then the conclusion of Theorem 3.2 is proved. Hence, we need only to consider those solution $(S(t), x(t))$ of system (1.1) such that $x(t)$ is oscillatory about m_3.

Let t_1 and t_2 be two large enough times such that $x(t_1) = x(t_2) = m_3$ and $x(t) < m_3$ for all $t \in (t_1, t_2)$. When $t_2 - t_1 \leq T_0$, since

$$x'(t) \geq -(D+\gamma)x(t) \quad \text{for all} \quad t \in (t_1, t_2),$$

integrating this inequality for any $t \in [t_1, t_2]$, we have

$$\begin{aligned} x(t) &\geq x(t_1) \exp\left\{\int\limits_{t_1}^{t} -(D+\gamma)\mathrm{d}v\right\} \\ &\geq m_3 \exp\{-(D+\gamma)T_0\} \\ &\triangleq m_2^*. \end{aligned} \tag{3.12}$$

Let $t_1 - t_2 > T_0$. For any $t \in [t_1, t_2]$, if $t \leq t_1 + T_0$, then according to the above discussing on the case of $t_2 - t_1 \leq T_0$, we also have inequality (3.12). Particularly, we obtain $x(t_1+T_0) \geq m_2^*$. Since $x(t) \leq m_3$ for all $t \in [t_1, t_2]$, from system (1.1) we have

$$\begin{aligned} S'(t) &\geq -S(t)\left(D + \frac{m_3\mu_m}{\delta K_m}\right), \quad && t \neq nT, \\ S(t^+) &= S(t) + DS^0, && t = nT. \end{aligned}$$

Hence, from the comparison theorem of impulsive differential equations we have $S(t) \geq y(t)$ for all $t \in [t_1, t_2]$, when $y(t)$ is the solution of Eq. 3.8 with initial condition $y(t_1^+) = S(t_1^+)$. From (3.10) we directly obtain

$$y(t) \geq y^*(t) - \frac{\varepsilon_0}{2} \quad \text{for all} \quad t \in [t_1+T_0, t_2].$$

Further, from (3.9) we also have

$$S(t) \geq u^*(t) - \varepsilon_0 \quad \text{for all} \quad t \in [t_1+T_0, t_2].$$

Therefore, from system (1.1) we further have

$$x'(t) \geq x(t)\left(\frac{\mu_m(S^*(t)-\varepsilon_0)}{K_m + S^*(t) - \varepsilon_0} - (D+\gamma)\right) \quad \text{for all} \quad t \in [t_0+T_0, t_2]. \tag{3.13}$$

For any $t \in [t_1+T_0, t_2]$, we choose an integer $p \geq 0$ such that $t \in [t_1+T_0+pT, t_1+T_0+(p+1)T]$. Integrating inequality (3.13) from t_1+T_0 to t, we can obtain

$$\begin{aligned} x(t) &= x(t_1+T_0)\exp\left\{\int_{t_1+T_0}^{t}\left(\frac{\mu_m(S^*(v)-\varepsilon_0)}{K_m+S^*(v)-\varepsilon_0} - (D+\gamma)\right)\mathrm{d}v\right\} \\ &\geq m_2^*\exp\left\{\int_{t_1+T_0}^{t_1+T_0+pT} + \int_{t_1+T_0+pT}^{t}\left(\frac{\mu_m(S^*(v)-\varepsilon_0)}{K_m+S(v)-\varepsilon_0} - (D+\gamma)\right)\mathrm{d}v\right\} \\ &\geq m_2^*\exp\left\{\int_{t_1+T_0+pt}^{t}\left(\frac{\mu_m(S^*(v)-\varepsilon_0)}{K_m+S^*(v)-\varepsilon_0} - (D+\gamma)\right)\mathrm{d}v\right\} \\ &\geq m_2^*\exp\{-hT\} \\ &\triangleq m_2 \end{aligned}$$

where

$$h = \sup_{t\geq 0}\left\{\frac{\mu_m(u^*(t)-\varepsilon_0)}{K_m+u^*(t)-\varepsilon_0} - (D+\gamma)\right\}.$$

From above discussions, we finally obtain

$$\liminf_{t\to\infty} x(t) \geq m_2,$$

and m_2 is independent of any solution $(S(t), x(t))$ of system (1.1). This completes the proof of Theorem 3.2. □

As a consequence of Theorems 3.1 and 3.2, we have the following corollary.

Corollary 3.1 *For system* (1.1)*, the following conclusions hold.*

(a) $(u^*(t), 0)$ *is globally attractive if and only if*

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D+\gamma) \right) \mathrm{d}t \leq 0.$$

(b) *System* (1.1) *is permanent if and only if*

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D+\gamma) \right) \mathrm{d}t > 0.$$

Now, we discuss the global attractivity of all positive solutions of system (1.1), we have the following result.

Theorem 3.3 *Suppose for system* (1.1)

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D+\gamma) \right) \mathrm{d}t > 0 \tag{3.14}$$

and

$$\frac{DK_m^2}{\delta(K_m + M)^2} - \gamma\left(\frac{1}{\delta} - b\right) > 0, \tag{3.15}$$

where $M = \frac{DS^0}{1-e^{-DT}}$. *Then for any two positive solutions* $(S_1(t), x_1(t))$ *and* $(S_2(t), x_2(t))$ *of system* (1.1)

$$\lim_{t\to\infty}(S_1(t) - S_2(t)) = 0, \quad \lim_{t\to\infty}(x_1(t) - x_2(t)) = 0.$$

Proof From condition (1), we directly have

$$\frac{DK_m}{\mu_m} > \frac{\gamma\left(\frac{1}{\delta} - b\right)\delta(K_m + M)^2}{\mu_m K_m}.$$

Hence, there is a constant $c > 0$ such that

$$D - \frac{c\mu_m}{K_m} > 0, \quad \frac{c\mu_m K_m}{\delta(K_m + M)^2} - \gamma\left(\frac{1}{\delta} - b\right) > 0.$$

Further, we can choose a constant $\varepsilon_0 > 0$ such that

$$D - \frac{c\mu_m}{K_m} > 0, \quad \frac{c\mu_m K_m}{\delta(K_m + M + \varepsilon_0)^2} - \gamma\left(\frac{1}{\delta} - b\right) > 0. \tag{3.16}$$

Let $V = S + \frac{1}{\delta}x$, then system (1.1) is equivalent to the following system

$$\begin{cases} V'(t) &= -DV(t) + \gamma\left(b - \frac{1}{\delta}\right)x(t), & t \neq nT, \\ x'(t) &= x(t)\left(\frac{\mu_m(V(t) - \frac{1}{\delta}x(t))}{K_m + V(t) - \frac{1}{\delta}x(t)} - (D + \gamma)\right), & t \neq nT, \\ V(t^+) &= V(t) + DS^0, & t = nT, \\ x(t^+) &= x(t), & t = nT. \end{cases} \tag{3.17}$$

Let $(S_1(t), x_1(t))$ and $(S_2(t), x_2(t))$ be any two positive solutions of system (1.1), from Lemma 2.2 and Theorem 3.2 we have that there is a constant $m > 0$ such that

$$m \le \liminf_{t\to\infty} S_i(t) \le \limsup_{t\to\infty} S_i(t) \le M$$

and

$$m \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le \delta M$$

for $i = 1, 2$. Hence, there exists a $T > 0$ such that

$$S_i(t) \le M + \varepsilon_0, \quad x_i(t) \ge \frac{1}{2}m \quad \text{for all} \quad t \ge T. \tag{3.18}$$

let $V_i(t) = S_i(t) + \frac{1}{\delta}x_i(t)\ (i = 1, 2)$, then $(V_i(t), x_i(t))$ is the solution of system (3.17). Define the Liapunov function as follows

$$U(t) = |V_1(t) - V_2(t)| + c|\ln x_1(t) - \ln x_2(t)|.$$

From (3.18) and the theorem of mean value we can obtain

$$|x_1(t) - x_2(t)| \ge \frac{1}{2}m|\ln x_1(t) - \ln x_2(t)| \tag{3.19}$$

and

$$\frac{\mu_m S_1(t)}{K_m + S_1(t)} - \frac{\mu_m S_1(t)}{K_m + S_1(t)} = \frac{\mu_m K_m}{(K_m + \xi(t))^2}(S_1(t) - S_2(t)), \tag{3.20}$$

for all $t \geq T$, where $\xi(t)$ is situated between $S_1(t)$ and $S_2(t)$.

Calculating the Dini derivative of $U(t)$, from (3.19) and (3.20) we obtain for any $t \geq 0$ and $t \neq nT$

$$\begin{aligned}
\dot{U}(t) = \operatorname{sign}(V_1(t) - V_2(t)) &\left(-D(V_1(t) - V_2(t)) + \gamma\left(b - \frac{1}{\delta}\right)(x_1(t) - x_2(t))\right) \\
&+ c\operatorname{sign}(x_1(t) - x_2(t))\left(\frac{\mu_m\left(V_1(t) - \frac{1}{\delta}x_1(t)\right)}{K_m + V_1(t) - \frac{1}{\delta}x_1(t)} - \frac{\mu_m\left(V_2(t) - \frac{1}{\delta}x_2(t)\right)}{K_m + V_2(t) - \frac{1}{\delta}x_2(t)}\right) \\
\leq & -D|V_1(t) - V_2(t)| + \gamma\left(\frac{1}{\delta} - b\right)|x_1(t) - x_2(t)| + c\operatorname{sign}(x_1(t) \\
& - x_2(t))\frac{\mu_m K_m}{(K_m + \xi(t))^2}\left(V_1(t) - V_2(t) - \frac{1}{\delta}(x_1(t) - x_2(t))\right) \\
\leq & -D|V_1(t) - V_2(t)| + \gamma\left(\frac{1}{\delta} - b\right)|x_1(t) - x_2(t)| + \frac{c\mu_m}{K_m}|V_1(t) - V_2(t)| \\
& - \frac{c\mu_m K_m}{\delta(K_m + M + \varepsilon_0)^2}|x_1(t) - x_2(t)| \leq \alpha U(t),
\end{aligned}$$

where

$$\alpha = \min\left\{D - \frac{c\mu}{K_m}, \frac{m}{2c}\left(\frac{c\mu_m K_m}{\delta(K_m + M + \varepsilon_0)^2} - \gamma\left(\frac{1}{\delta} - b\right)\right)\right\}$$

and from (3.16) we obtain $\alpha > 0$. On the other hand, we directly

$$U(t^+) = U(t) \quad \text{for all} \quad t = nT,\ n \in N.$$

Hence, for any $t > 0$ we have

$$U(t) \leq U(0)\exp(-\alpha t).$$

Consequently, $\lim_{t\to\infty} U(t) = 0$. From this, we finally obtain

$$\lim_{t\to\infty}(S_1(t) - S_2(t)) = 0, \quad \lim_{t\to\infty}(x_1(t) - x_2(t)) = 0.$$

This completes the proof of Theorem 3.3. □

When $\gamma = 0$, then system (1.1) degenerate to the following system without nutrient recycling

$$\begin{aligned} S'(t) &= -DS(t) - \tfrac{\mu_m S(t)x(t)}{\delta(K_m+S(t))}, && t \neq nT, \\ x'(t) &= \tfrac{\mu_m S(t)x(t)}{K_m+S(t)} - Dx(t), && t \neq nT, \\ S(t^+) &= S(t) + DS^0, && t = nT, \\ x(t^+) &= x(t), && t = nT. \end{aligned} \tag{3.21}$$

We see that condition (3.15) always holds. Hence, from Theorems 3.1–3.3 we can obtain the following results.

Corollary 3.2 *For system* (3.21)*, the following conclusions hold.*

(a) $(u^*(t), 0)$ *is globally attractive if and only if*

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - D\right) \mathrm{d}t \leq 0.$$

(b) *System* (3.21) *is permanent and globally attractive if and only if*

$$\int_0^T \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - D\right) \mathrm{d}t > 0.$$

Remark 3.1 Obviously, Corollary 3.2 is an very good improvement and extension of the corresponding results given in [19], see Theorems 3.1–3.3 in [19].

Remark 3.2 When $\gamma > 0$ and $b = \frac{1}{\delta}$, we see that condition (3.15) also holds. Hence, system (1.1) is globally attractive so long as condition (3.14) holds. Therefore, an important and interesting open problem is proposed here, that is, when $\gamma > 0$ and $b < \frac{1}{\delta}$ whether system (1.1) also is globally attractive so long as condition (3.14) holds.

4 An example

In this section, we will give an example to show that if the condition (3.14) holds, but condition (3.15) does not hold, then system (1.1) still is globally asymptotically stable.

We consider the following special case of system (1.1)

$$\begin{aligned} S'(t) &= -S(t) - \frac{12S(t)x(t)}{0.7(8+S(t))} + 0.6 \times 0.3x(t), && t \neq 2n, \\ x'(t) &= \frac{12S(t)x(t)}{8+S(t)} - (1+0.3)x(t), && t \neq 2n, \\ S(t^+) &= S(t) + 10, && t = 2n, \\ x(t^+) &= x(t), && t = 2n, \end{aligned} \tag{3.22}$$

that is, in system (1.1) we take

$$D = 1,\ \mu_m = 12,\ K_m = 8,\ \delta = 0.7,\ b = 0.6,\ \gamma = 0.3,\ S^0 = 10,\ T = 2.$$

By calculating, we obtain

$$\delta(K_m + M)^2 \gamma \left(\frac{1}{\delta} - b\right) = 66.6065, \quad D(K_m)^2 = 64,$$
$$u^*(t) = \frac{10e^{-(t-2n)}}{1 - e^{-2}}, \quad t \in (2n,\ 2(n+1)],\ n \in N$$

and

$$\int_0^2 \left(\frac{\mu_m u^*(t)}{K_m + u^*(t)} - (D + \gamma)\right) \mathrm{d}t = 5.9875.$$

Therefore, conditions (3.14) holds, but (3.15) does not hold. But, we choose initial value

$$(S_0, x_0) = (1, 3.5), (3, 3.0), (5, 2.5), (7, 2.0), (9, 1.5), (11, 1.0), (13, 0.5),$$

respectively, then from the numerical simulation (see Figs. 1, 2) we see that there exists a unique positive T-periodic solution $(S^*(t), x^*(t))$ of system (3.22) such that any solution $(S(t), x(t))$ of system (3.22) with initial value (S_0, x_0) tend to $(S^*(t), x^*(t))$ as $t \to \infty$. Therefore, we can guess that if only condition (3.14) holds then system (3.22) has a unique positive T-periodic solution which is globally attractive

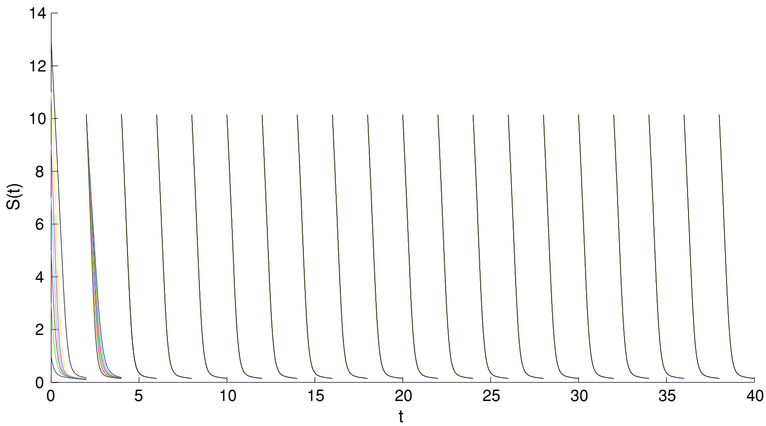

Fig. 1 Time series of $S(t)$

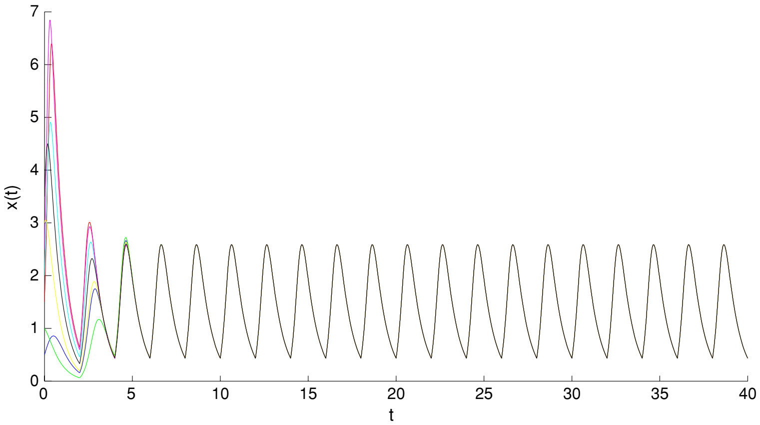

Fig. 2 Time series of $x(t)$

Acknowledgments Supported by The National Natural Science Foundation of P. R. China (60764003), the major project of The Ministry of Education of P. R. China (207130) and the scientific research programmes of colleges in Xinjiang (XJEDU2007G01, XJEDU2006I05).

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